

# LINEAR ALGEBRA AND ITS APPLICATIONS

FIFTH EDITION

DAVID C. LAY • STEVEN R. LAY • JUDI J. McDONALD



FIFTH EDITION

# Linear Algebra and Its Applications

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with

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and

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# About the Author

David C. Lay holds a B.A. from Aurora University (Illinois), and an M.A. and Ph.D. from the University of California at Los Angeles. David Lay has been an educator and research mathematician since 1966, mostly at the University of Maryland, College Park. He has also served as a visiting professor at the University of Amsterdam, the Free University in Amsterdam, and the University of Kaiserslautern, Germany. He has published more than 30 research articles on functional analysis and linear algebra.

As a founding member of the NSF-sponsored Linear Algebra Curriculum Study Group, David Lay has been a leader in the current movement to modernize the linear algebra curriculum. Lay is also a coauthor of several mathematics texts, including *Introduction to Functional Analysis* with Angus E. Taylor, *Calculus and Its Applications*, with L. J. Goldstein and D. I. Schneider, and *Linear Algebra Gems—Assets for Undergraduate Mathematics*, with D. Carlson, C. R. Johnson, and A. D. Porter.

David Lay has received four university awards for teaching excellence, including, in 1996, the title of Distinguished Scholar–Teacher of the University of Maryland. In 1994, he was given one of the Mathematical Association of America’s Awards for Distinguished College or University Teaching of Mathematics. He has been elected by the university students to membership in Alpha Lambda Delta National Scholastic Honor Society and Golden Key National Honor Society. In 1989, Aurora University conferred on him the Outstanding Alumnus award. David Lay is a member of the American Mathematical Society, the Canadian Mathematical Society, the International Linear Algebra Society, the Mathematical Association of America, Sigma Xi, and the Society for Industrial and Applied Mathematics. Since 1992, he has served several terms on the national board of the Association of Christians in the Mathematical Sciences.

*To my wife, Lillian, and our children,  
Christina, Deborah, and Melissa, whose  
support, encouragement, and faithful  
prayers made this book possible.*

David C. Lay

# Joining the Authorship on the Fifth Edition

## Steven R. Lay

Steven R. Lay began his teaching career at Aurora University (Illinois) in 1971, after earning an M.A. and a Ph.D. in mathematics from the University of California at Los Angeles. His career in mathematics was interrupted for eight years while serving as a missionary in Japan. Upon his return to the States in 1998, he joined the mathematics faculty at Lee University (Tennessee) and has been there ever since. Since then he has supported his brother David in refining and expanding the scope of this popular linear algebra text, including writing most of Chapters 8 and 9. Steven is also the author of three college-level mathematics texts: *Convex Sets and Their Applications*, *Analysis with an Introduction to Proof*, and *Principles of Algebra*.

In 1985, Steven received the Excellence in Teaching Award at Aurora University. He and David, and their father, Dr. L. Clark Lay, are all distinguished mathematicians, and in 1989 they jointly received the Outstanding Alumnus award from their alma mater, Aurora University. In 2006, Steven was honored to receive the Excellence in Scholarship Award at Lee University. He is a member of the American Mathematical Society, the Mathematics Association of America, and the Association of Christians in the Mathematical Sciences.

## Judi J. McDonald

Judi J. McDonald joins the authorship team after working closely with David on the fourth edition. She holds a B.Sc. in Mathematics from the University of Alberta, and an M.A. and Ph.D. from the University of Wisconsin. She is currently a professor at Washington State University. She has been an educator and research mathematician since the early 90s. She has more than 35 publications in linear algebra research journals. Several undergraduate and graduate students have written projects or theses on linear algebra under Judi's supervision. She has also worked with the mathematics outreach project Math Central <http://mathcentral.uregina.ca/> and continues to be passionate about mathematics education and outreach.

Judi has received three teaching awards: two Inspiring Teaching awards at the University of Regina, and the Thomas Lutz College of Arts and Sciences Teaching Award at Washington State University. She has been an active member of the International Linear Algebra Society and the Association for Women in Mathematics throughout her career and has also been a member of the Canadian Mathematical Society, the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

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# Preface

The response of students and teachers to the first four editions of *Linear Algebra and Its Applications* has been most gratifying. This *Fifth Edition* provides substantial support both for teaching and for using technology in the course. As before, the text provides a modern elementary introduction to linear algebra and a broad selection of interesting applications. The material is accessible to students with the maturity that should come from successful completion of two semesters of college-level mathematics, usually calculus.

The main goal of the text is to help students master the basic concepts and skills they will use later in their careers. The topics here follow the recommendations of the Linear Algebra Curriculum Study Group, which were based on a careful investigation of the real needs of the students and a consensus among professionals in many disciplines that use linear algebra. We hope this course will be one of the most useful and interesting mathematics classes taken by undergraduates.

## WHAT'S NEW IN THIS EDITION

The main goals of this revision were to update the exercises, take advantage of improvements in technology, and provide more support for conceptual learning.

1. Support for the *Fifth Edition* is offered through MyMathLab. MyMathLab, from Pearson, is the world's leading online resource in mathematics, integrating interactive homework, assessment, and media in a flexible, easy-to-use format. Students submit homework online for instantaneous feedback, support, and assessment. This system works particularly well for computation-based skills. Many additional resources are also provided through the MyMathLab web site.
2. The *Fifth Edition* of the text is available in an interactive electronic format. Using the CDF player, a free Mathematica player available from Wolfram, students can interact with figures and experiment with matrices by looking at numerous examples with just the click of a button. The geometry of linear algebra comes alive through these interactive figures. Students are encouraged to develop conjectures through experimentation and then verify that their observations are correct by examining the relevant theorems and their proofs. The resources in the interactive version of the text give students the opportunity to play with mathematical objects and ideas much as we do with our own research. Files for Wolfram CDF Player are also available for classroom presentations.
3. The *Fifth Edition* includes additional support for concept- and proof-based learning. Conceptual Practice Problems and their solutions have been added so that most sections now have a proof- or concept-based example for students to review. Additional guidance has also been added to some of the proofs of theorems in the body of the textbook.

4. More than 25 percent of the exercises are new or updated, especially the computational exercises. The exercise sets remain one of the most important features of this book, and these new exercises follow the same high standard of the exercise sets from the past four editions. They are crafted in a way that reflects the substance of each of the sections they follow, developing the students' confidence while challenging them to practice and generalize the new ideas they have encountered.

## DISTINCTIVE FEATURES

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### Early Introduction of Key Concepts

Many fundamental ideas of linear algebra are introduced within the first seven lectures, in the concrete setting of  $\mathbb{R}^n$ , and then gradually examined from different points of view. Later generalizations of these concepts appear as natural extensions of familiar ideas, visualized through the geometric intuition developed in Chapter 1. A major achievement of this text is that the level of difficulty is fairly even throughout the course.

### A Modern View of Matrix Multiplication

Good notation is crucial, and the text reflects the way scientists and engineers actually use linear algebra in practice. The definitions and proofs focus on the columns of a matrix rather than on the matrix entries. A central theme is to view a matrix–vector product  $A\mathbf{x}$  as a linear combination of the columns of  $A$ . This modern approach simplifies many arguments, and it ties vector space ideas into the study of linear systems.

### Linear Transformations

Linear transformations form a “thread” that is woven into the fabric of the text. Their use enhances the geometric flavor of the text. In Chapter 1, for instance, linear transformations provide a dynamic and graphical view of matrix–vector multiplication.

### Eigenvalues and Dynamical Systems

Eigenvalues appear fairly early in the text, in Chapters 5 and 7. Because this material is spread over several weeks, students have more time than usual to absorb and review these critical concepts. Eigenvalues are motivated by and applied to discrete and continuous dynamical systems, which appear in Sections 1.10, 4.8, and 4.9, and in five sections of Chapter 5. Some courses reach Chapter 5 after about five weeks by covering Sections 2.8 and 2.9 instead of Chapter 4. These two optional sections present all the vector space concepts from Chapter 4 needed for Chapter 5.

### Orthogonality and Least-Squares Problems

These topics receive a more comprehensive treatment than is commonly found in beginning texts. The Linear Algebra Curriculum Study Group has emphasized the need for a substantial unit on orthogonality and least-squares problems, because orthogonality plays such an important role in computer calculations and numerical linear algebra and because inconsistent linear systems arise so often in practical work.

## PEDAGOGICAL FEATURES

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### Applications

A broad selection of applications illustrates the power of linear algebra to explain fundamental principles and simplify calculations in engineering, computer science, mathematics, physics, biology, economics, and statistics. Some applications appear in separate sections; others are treated in examples and exercises. In addition, each chapter opens with an introductory vignette that sets the stage for some application of linear algebra and provides a motivation for developing the mathematics that follows. Later, the text returns to that application in a section near the end of the chapter.

### A Strong Geometric Emphasis

Every major concept in the course is given a geometric interpretation, because many students learn better when they can visualize an idea. There are substantially more drawings here than usual, and some of the figures have never before appeared in a linear algebra text. Interactive versions of these figures, and more, appear in the electronic version of the textbook.

### Examples

This text devotes a larger proportion of its expository material to examples than do most linear algebra texts. There are more examples than an instructor would ordinarily present in class. But because the examples are written carefully, with lots of detail, students can read them on their own.

### Theorems and Proofs

Important results are stated as theorems. Other useful facts are displayed in tinted boxes, for easy reference. Most of the theorems have formal proofs, written with the beginner student in mind. In a few cases, the essential calculations of a proof are exhibited in a carefully chosen example. Some routine verifications are saved for exercises, when they will benefit students.

### Practice Problems

A few carefully selected Practice Problems appear just before each exercise set. Complete solutions follow the exercise set. These problems either focus on potential trouble spots in the exercise set or provide a “warm-up” for the exercises, and the solutions often contain helpful hints or warnings about the homework.

### Exercises

The abundant supply of exercises ranges from routine computations to conceptual questions that require more thought. A good number of innovative questions pinpoint conceptual difficulties that we have found on student papers over the years. Each exercise set is carefully arranged in the same general order as the text; homework assignments are readily available when only part of a section is discussed. A notable feature of the exercises is their numerical simplicity. Problems “unfold” quickly, so students spend little time on numerical calculations. The exercises concentrate on teaching understanding rather than mechanical calculations. The exercises in the *Fifth Edition* maintain the integrity of the exercises from previous editions, while providing fresh problems for students and instructors.

Exercises marked with the symbol [M] are designed to be worked with the aid of a “Matrix program” (a computer program, such as MATLAB<sup>®</sup>, Maple<sup>™</sup>, Mathematica<sup>®</sup>,

MathCad<sup>®</sup>, or Derive<sup>™</sup>, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments).

## True/False Questions

To encourage students to read all of the text and to think critically, we have developed 300 simple true/false questions that appear in 33 sections of the text, just after the computational problems. They can be answered directly from the text, and they prepare students for the conceptual problems that follow. Students appreciate these questions—after they get used to the importance of reading the text carefully. Based on class testing and discussions with students, we decided not to put the answers in the text. (The *Study Guide* tells the students where to find the answers to the odd-numbered questions.) An additional 150 true/false questions (mostly at the ends of chapters) test understanding of the material. The text does provide simple T/F answers to most of these questions, but it omits the justifications for the answers (which usually require some thought).

## Writing Exercises

An ability to write coherent mathematical statements in English is essential for all students of linear algebra, not just those who may go to graduate school in mathematics. The text includes many exercises for which a written justification is part of the answer. Conceptual exercises that require a short proof usually contain hints that help a student get started. For all odd-numbered writing exercises, either a solution is included at the back of the text or a hint is provided and the solution is given in the *Study Guide*, described below.

## Computational Topics

The text stresses the impact of the computer on both the development and practice of linear algebra in science and engineering. Frequent Numerical Notes draw attention to issues in computing and distinguish between theoretical concepts, such as matrix inversion, and computer implementations, such as LU factorizations.

## WEB SUPPORT

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### MyMathLab—Online Homework and Resources

Support for the *Fifth Edition* is offered through MyMathLab ([www.mymathlab.com](http://www.mymathlab.com)). MyMathLab from Pearson is the world's leading online resource in mathematics, integrating interactive homework, assessment, and media in a flexible, easy-to-use format. MyMathLab contains hundreds of algorithmically generated exercises that mirror those in the textbook. Students submit homework online for instantaneous feedback, support, and assessment. This system works particularly well for supporting computation-based skills. Many additional resources are also provided through the MyMathLab web site.

### Interactive Textbook

The *Fifth Edition* of the text is available in an interactive electronic format within MyMathLab. Using Wolfram CDF Player, a free Mathematica player available from Wolfram ([www.wolfram.com/player](http://www.wolfram.com/player)), students can interact with figures and experiment with matrices by looking at numerous examples. The geometry of linear algebra comes alive through these interactive figures. Students are encouraged to develop conjectures

through experimentation, then verify that their observations are correct by examining the relevant theorems and their proofs. The resources in the interactive version of the text give students the opportunity to interact with mathematical objects and ideas much as we do with our own research.

This web site at [www.pearsonhighered.com/lay](http://www.pearsonhighered.com/lay) contains all of the support material referenced below. These materials are also available within MyMathLab.

## Review Material

Review sheets and practice exams (with solutions) cover the main topics in the text. They come directly from courses we have taught in the past years. Each review sheet identifies key definitions, theorems, and skills from a specified portion of the text.

## Applications by Chapters

The web site contains seven Case Studies, which expand topics introduced at the beginning of each chapter, adding real-world data and opportunities for further exploration. In addition, more than 20 Application Projects either extend topics in the text or introduce new applications, such as cubic splines, airline flight routes, dominance matrices in sports competition, and error-correcting codes. Some mathematical applications are integration techniques, polynomial root location, conic sections, quadric surfaces, and extrema for functions of two variables. Numerical linear algebra topics, such as condition numbers, matrix factorizations, and the QR method for finding eigenvalues, are also included. Woven into each discussion are exercises that may involve large data sets (and thus require technology for their solution).

## Getting Started with Technology


If your course includes some work with MATLAB, Maple, Mathematica, or TI calculators, the Getting Started guides provide a “quick start guide” for students.

Technology-specific projects are also available to introduce students to software and calculators. They are available on [www.pearsonhighered.com/lay](http://www.pearsonhighered.com/lay) and within MyMathLab. Finally, the Study Guide provides introductory material for first-time technology users.

## Data Files

Hundreds of files contain data for about 900 numerical exercises in the text, Case Studies, and Application Projects. The data are available in a variety of formats—for MATLAB, Maple, Mathematica, and the Texas Instruments graphing calculators. By allowing students to access matrices and vectors for a particular problem with only a few keystrokes, the data files eliminate data entry errors and save time on homework. These data files are available for download at [www.pearsonhighered.com/lay](http://www.pearsonhighered.com/lay) and MyMathLab.


## Projects

Exploratory projects for Mathematica,<sup>TM</sup> Maple, and MATLAB invite students to discover basic mathematical and numerical issues in linear algebra. Written by experienced faculty members, these projects are referenced by the icon  at appropriate points in the text. The projects explore fundamental concepts such as the column space, diagonalization, and orthogonal projections; several projects focus on numerical issues such as flops, iterative methods, and the SVD; and a few projects explore applications such as Lagrange interpolation and Markov chains.

## SUPPLEMENTS

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### Study Guide

A printed version of the *Study Guide* is available at low cost. It is also available electronically within MyMathLab. The *Guide* is designed to be an integral part of the course. The icon  in the text directs students to special subsections of the *Guide* that suggest how to master key concepts of the course. The *Guide* supplies a detailed solution to every third odd-numbered exercise, which allows students to check their work. A complete explanation is provided whenever an odd-numbered writing exercise has only a “Hint” in the answers. Frequent “Warnings” identify common errors and show how to prevent them. MATLAB boxes introduce commands as they are needed. Appendixes in the *Study Guide* provide comparable information about Maple, Mathematica, and TI graphing calculators (ISBN: 0-321-98257-6).

### Instructor’s Edition

For the convenience of instructors, this special edition includes brief answers to all exercises. A *Note to the Instructor* at the beginning of the text provides a commentary on the design and organization of the text, to help instructors plan their courses. It also describes other support available for instructors (ISBN: 0-321-98261-4).

### Instructor’s Technology Manuals

Each manual provides detailed guidance for integrating a specific software package or graphing calculator throughout the course, written by faculty who have already used the technology with this text. The following manuals are available to qualified instructors through the Pearson Instructor Resource Center, [www.pearsonhighered.com/irc](http://www.pearsonhighered.com/irc) and MyMathLab: MATLAB (ISBN: 0-321-98985-6), Maple (ISBN: 0-134-04726-5), Mathematica (ISBN: 0-321-98975-9), and TI-83+/89 (ISBN: 0-321-98984-8).

### Instructor’s Solutions Manual

The *Instructor’s Solutions Manual* (ISBN 0-321-98259-2) contains detailed solutions for all exercises, along with teaching notes for many sections. The manual is available electronically for download in the Instructor Resource Center ([www.pearsonhighered.com/lay](http://www.pearsonhighered.com/lay)) and MyMathLab.

### PowerPoint® Slides and Other Teaching Tools

A brisk pace at the beginning of the course helps to set the tone for the term. To get quickly through the first two sections in fewer than two lectures, consider using PowerPoint® slides (ISBN 0-321-98264-9). They permit you to focus on the process of row reduction rather than to write many numbers on the board. Students can receive a condensed version of the notes, with occasional blanks to fill in during the lecture. (Many students respond favorably to this gesture.) The PowerPoint slides are available for 25 core sections of the text. In addition, about 75 color figures from the text are available as PowerPoint slides. The PowerPoint slides are available for download at [www.pearsonhighered.com/irc](http://www.pearsonhighered.com/irc). Interactive figures are available as Wolfram CDF Player files for classroom demonstrations. These files provide the instructor with the opportunity to bring the geometry alive and to encourage students to make conjectures by looking at numerous examples. The files are available exclusively within MyMathLab.

## TestGen

TestGen ([www.pearsonhighered.com/testgen](http://www.pearsonhighered.com/testgen)) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple, but equivalent, versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions. The software and test bank are available for download from Pearson Education's online catalog. (ISBN: 0-321-98260-6)

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I want to thank Israel Gohberg and Robert Ellis for more than fifteen years of research collaboration, which greatly shaped my view of linear algebra. And it has been a privilege to be a member of the Linear Algebra Curriculum Study Group along with David Carlson, Charles Johnson, and Duane Porter. Their creative ideas about teaching linear algebra have influenced this text in significant ways.

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David C. Lay

It has been a privilege to work on this new *Fifth Edition* of Professor David Lay's linear algebra book. In making this revision, we have attempted to maintain the basic approach and the clarity of style that has made earlier editions popular with students and faculty.

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We thank Kristina Evans and Phil Oslin for their work in setting up and maintaining the online homework to accompany the text in MyMathLab, and for continuing to work with us to improve it. The reviews of the online homework done by Joan Saniuk, Robert Pierce, Doron Lubinsky and Adriana Corinaldesi were greatly appreciated. We also thank the faculty at University of California Santa Barbara, University of Alberta, and Georgia Institute of Technology for their feedback on the MyMathLab course.

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Steven R. Lay and Judi J. McDonald

# A Note to Students

This course is potentially the most interesting and worthwhile undergraduate mathematics course you will complete. In fact, some students have written or spoken to us after graduation and said that they still use this text occasionally as a reference in their careers at major corporations and engineering graduate schools. The following remarks offer some practical advice and information to help you master the material and enjoy the course.

In linear algebra, the *concepts* are as important as the *computations*. The simple numerical exercises that begin each exercise set only help you check your understanding of basic procedures. Later in your career, computers will do the calculations, but you will have to choose the calculations, know how to interpret the results, and then explain the results to other people. For this reason, many exercises in the text ask you to explain or justify your calculations. A written explanation is often required as part of the answer. For odd-numbered exercises, you will find either the desired explanation or at least a good hint. You must avoid the temptation to look at such answers before you have tried to write out the solution yourself. Otherwise, you are likely to think you understand something when in fact you do not.

To master the concepts of linear algebra, you will have to read and reread the text carefully. New terms are in boldface type, sometimes enclosed in a definition box. A glossary of terms is included at the end of the text. Important facts are stated as theorems or are enclosed in tinted boxes, for easy reference. We encourage you to read the first five pages of the Preface to learn more about the structure of this text. This will give you a framework for understanding how the course may proceed.

In a practical sense, linear algebra is a language. You must learn this language the same way you would a foreign language—with daily work. Material presented in one section is not easily understood unless you have thoroughly studied the text and worked the exercises for the preceding sections. Keeping up with the course will save you lots of time and distress!


## Numerical Notes

We hope you read the Numerical Notes in the text, even if you are not using a computer or graphing calculator with the text. In real life, most applications of linear algebra involve numerical computations that are subject to some numerical error, even though that error may be extremely small. The Numerical Notes will warn you of potential difficulties in using linear algebra later in your career, and if you study the notes now, you are more likely to remember them later.

If you enjoy reading the Numerical Notes, you may want to take a course later in numerical linear algebra. Because of the high demand for increased computing power, computer scientists and mathematicians work in numerical linear algebra to develop faster and more reliable algorithms for computations, and electrical engineers design faster and smaller computers to run the algorithms. This is an exciting field, and your first course in linear algebra will help you prepare for it.



## Study Guide

To help you succeed in this course, we suggest that you purchase the *Study Guide* ([www.mypearsonstore.com](http://www.mypearsonstore.com); 0-321-98257-6). It is available electronically within MyMathLab. Not only will it help you learn linear algebra, it also will show you how to study mathematics. At strategic points in your textbook, the icon  will direct you to special subsections in the *Study Guide* entitled “Mastering Linear Algebra Concepts.” There you will find suggestions for constructing effective review sheets of key concepts. The act of preparing the sheets is one of the secrets to success in the course, because you will construct *links between ideas*. These links are the “glue” that enables you to build a solid foundation for learning and *remembering* the main concepts in the course.

The *Study Guide* contains a detailed solution to every third odd-numbered exercise, plus solutions to all odd-numbered writing exercises for which only a hint is given in the Answers section of this book. The *Guide* is separate from the text because you must learn to write solutions by yourself, without much help. (We know from years of experience that easy access to solutions in the back of the text slows the mathematical development of most students.) The *Guide* also provides warnings of common errors and helpful hints that call attention to key exercises and potential exam questions.

If you have access to technology—MATLAB, Maple, Mathematica, or a TI graphing calculator—you can save many hours of homework time. The *Study Guide* is your “lab manual” that explains how to use each of these matrix utilities. It introduces new commands when they are needed. You can download from the web site [www.pearsonhighered.com/lay](http://www.pearsonhighered.com/lay) the data for more than 850 exercises in the text. (With a few keystrokes, you can display any numerical homework problem on your screen.) Special matrix commands will perform the computations for you!

What you do in your first few weeks of studying this course will set your pattern for the term and determine how well you finish the course. Please read “How to Study Linear Algebra” in the *Study Guide* as soon as possible. Many students have found the strategies there very helpful, and we hope you will, too.

# 1

## Linear Equations in Linear Algebra

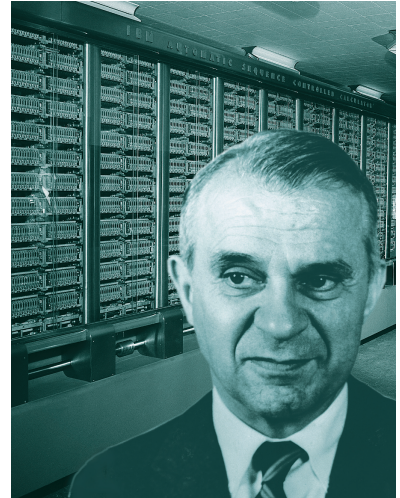
### INTRODUCTORY EXAMPLE

### Linear Models in Economics and Engineering

It was late summer in 1949. Harvard Professor Wassily Leontief was carefully feeding the last of his punched cards into the university's Mark II computer. The cards contained information about the U.S. economy and represented a summary of more than 250,000 pieces of information produced by the U.S. Bureau of Labor Statistics after two years of intensive work. Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns, Leontief had distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort, and he was anxious to see how long the computer would take to solve the problem. The Mark II hummed and blinked for 56 hours before finally producing a solution. We will discuss the nature of this solution in Sections 1.6 and 2.6.

Leontief, who was awarded the 1973 Nobel Prize in Economic Science, opened the door to a new era in mathematical modeling in economics. His efforts



at Harvard in 1949 marked one of the first significant uses of computers to analyze what was then a large-scale mathematical model. Since that time, researchers in many other fields have employed computers to analyze mathematical models. Because of the massive amounts of data involved, the models are usually *linear*; that is, they are described by *systems of linear equations*.

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Scientists and engineers now work on problems far more complex than even dreamed possible a few decades ago. Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate mathematics subject! The material in this text provides the foundation for further work in many interesting areas. Here are a few possibilities; others will be described later.

- *Oil exploration.* When a ship searches for offshore oil deposits, its computers solve thousands of separate systems of linear equations *every day*.

The seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface rocks and are measured by geophones attached to mile-long cables behind the ship.

- *Linear programming.* Many important management decisions today are made on the basis of linear programming models that use hundreds of variables. The airline industry, for instance, employs linear

programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.

- *Electrical networks.* Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.

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Systems of linear equations lie at the heart of linear algebra, and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. Sections 1.1 and 1.2 present a systematic method for solving systems of linear equations. This algorithm will be used for computations throughout the text. Sections 1.3 and 1.4 show how a system of linear equations is equivalent to a *vector equation* and to a *matrix equation*. This equivalence will reduce problems involving linear combinations of vectors to questions about systems of linear equations. The fundamental concepts of spanning, linear independence, and linear transformations, studied in the second half of the chapter, will play an essential role throughout the text as we explore the beauty and power of linear algebra.

## 1.1 SYSTEMS OF LINEAR EQUATIONS

A **linear equation** in the variables  $x_1, \dots, x_n$  is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (1)$$

where  $b$  and the **coefficients**  $a_1, \dots, a_n$  are real or complex numbers, usually known in advance. The subscript  $n$  may be any positive integer. In textbook examples and exercises,  $n$  is normally between 2 and 5. In real-life problems,  $n$  might be 50 or 5000, or even larger.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of  $x_1x_2$  in the first equation and  $\sqrt{x_1}$  in the second.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say,  $x_1, \dots, x_n$ . An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 &\quad - 4x_3 = -7 \end{aligned} \quad (2)$$

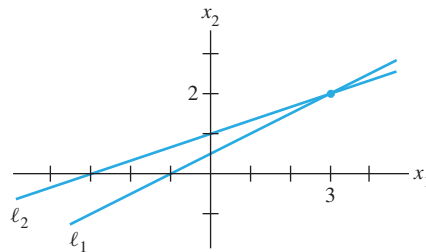
A **solution** of the system is a list  $(s_1, s_2, \dots, s_n)$  of numbers that makes each equation a true statement when the values  $s_1, \dots, s_n$  are substituted for  $x_1, \dots, x_n$ , respectively. For instance,  $(5, 6.5, 3)$  is a solution of system (2) because, when these values are substituted in (2) for  $x_1, x_2, x_3$ , respectively, the equations simplify to  $8 = 8$  and  $-7 = -7$ .

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned}x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3\end{aligned}$$

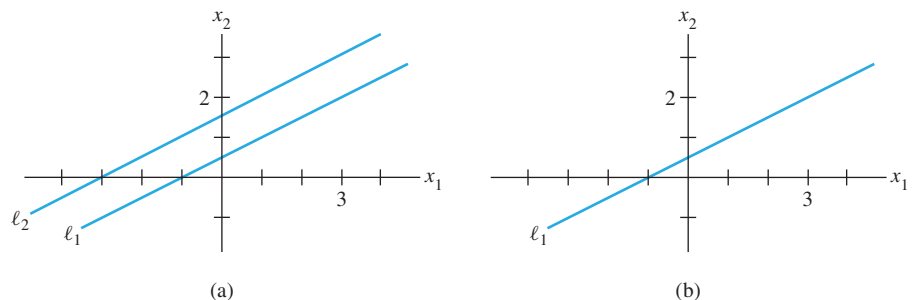
The graphs of these equations are lines, which we denote by  $\ell_1$  and  $\ell_2$ . A pair of numbers  $(x_1, x_2)$  satisfies *both* equations in the system if and only if the point  $(x_1, x_2)$  lies on both  $\ell_1$  and  $\ell_2$ . In the system above, the solution is the single point  $(3, 2)$ , as you can easily verify. See Figure 1.



**FIGURE 1** Exactly one solution.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence “intersect” at every point on the line. Figure 2 shows the graphs that correspond to the following systems:

$$\begin{array}{ll} \text{(a)} & x_1 - 2x_2 = -1 \\ & -x_1 + 2x_2 = 3 \\ \text{(b)} & x_1 - 2x_2 = -1 \\ & -x_1 + 2x_2 = 1 \end{array}$$



**FIGURE 2** (a) No solution. (b) Infinitely many solutions.

Figures 1 and 2 illustrate the following general fact about linear systems, to be verified in Section 1.2.

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

## Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\5x_1 - 5x_3 &= 10\end{aligned}\tag{3}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system (3), and

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}\tag{4}$$

is called the **augmented matrix** of the system. (The second row here contains a zero because the second equation could be written as  $0 \cdot x_1 + 2x_2 - 8x_3 = 8$ .) An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

The **size** of a matrix tells how many rows and columns it has. The augmented matrix (4) above has 3 rows and 4 columns and is called a  $3 \times 4$  (read “3 by 4”) matrix. If  $m$  and  $n$  are positive integers, an  **$m \times n$  matrix** is a rectangular array of numbers with  $m$  rows and  $n$  columns. (The number of rows always comes first.) Matrix notation will simplify the calculations in the examples that follow.

## Solving a Linear System

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is *to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve*.

Roughly speaking, use the  $x_1$  term in the first equation of a system to eliminate the  $x_1$  terms in the other equations. Then use the  $x_2$  term in the second equation to eliminate the  $x_2$  terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations.

Three basic operations are used to simplify a linear system: Replace one equation by the sum of itself and a multiple of another equation, interchange two equations, and multiply all the terms in an equation by a nonzero constant. After the first example, you will see why these three operations do not change the solution set of the system.

**EXAMPLE 1** Solve system (3).

**SOLUTION** The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison:

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & & \\ 2x_2 - 8x_3 = 8 & & \\ 5x_1 - 5x_3 = 10 & & \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Keep  $x_1$  in the first equation and eliminate it from the other equations. To do so, add  $-5$  times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{rcl} -5 \cdot [\text{equation 1}] & -5x_1 + 10x_2 - 5x_3 = 0 & \\ + [\text{equation 3}] & 5x_1 - 5x_3 = 10 & \\ \hline [\text{new equation 3}] & 10x_2 - 10x_3 = 10 & \end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & & \\ 2x_2 - 8x_3 = 8 & & \\ 10x_2 - 10x_3 = 10 & & \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Now, multiply equation 2 by  $\frac{1}{2}$  in order to obtain 1 as the coefficient for  $x_2$ . (This calculation will simplify the arithmetic in the next step.)

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & & \\ x_2 - 4x_3 = 4 & & \\ 10x_2 - 10x_3 = 10 & & \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Use the  $x_2$  in equation 2 to eliminate the  $10x_2$  in equation 3. The “mental” computation is

$$\begin{array}{rcl} -10 \cdot [\text{equation 2}] & -10x_2 + 40x_3 = -40 & \\ + [\text{equation 3}] & 10x_2 - 10x_3 = 10 & \\ \hline [\text{new equation 3}] & 30x_3 = -30 & \end{array}$$

The result of this calculation is written in place of the previous third equation (row):

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & & \\ x_2 - 4x_3 = 4 & & \\ 30x_3 = -30 & & \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Now, multiply equation 3 by  $\frac{1}{30}$  in order to obtain 1 as the coefficient for  $x_3$ . (This calculation will simplify the arithmetic in the next step.)

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & & \\ x_2 - 4x_3 = 4 & & \\ x_3 = -1 & & \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The new system has a *triangular* form (the intuitive term *triangular* will be replaced by a precise term in the next section):

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & & \\ x_2 - 4x_3 = 4 & & \\ x_3 = -1 & & \end{array} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Eventually, you want to eliminate the  $-2x_2$  term from equation 1, but it is more efficient to use the  $x_3$  in equation 3 first, to eliminate the  $-4x_3$  and  $+x_3$  terms in equations 2 and 1. The two “mental” calculations are

$$\begin{array}{rcl} 4 \cdot [\text{equation 3}] & 4x_3 = -4 & -1 \cdot [\text{equation 3}] & -x_3 = 1 \\ + [\text{equation 2}] & \underline{x_2 - 4x_3 = 4} & + [\text{equation 1}] & \underline{x_1 - 2x_2 + x_3 = 0} \\ \hline [\text{new equation 2}] & x_2 = 0 & [\text{new equation 1}] & x_1 - 2x_2 = 1 \end{array}$$

It is convenient to combine the results of these two operations:

$$\begin{array}{rcl} x_1 - 2x_2 & = & 1 \\ x_2 & = & 0 \\ x_3 & = & -1 \end{array} \quad \left[ \begin{array}{cccc} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Now, having cleaned out the column above the  $x_3$  in equation 3, move back to the  $x_2$  in equation 2 and use it to eliminate the  $-2x_2$  above it. Because of the previous work with  $x_3$ , there is now no arithmetic involving  $x_3$  terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 0 \\ x_3 & = & -1 \end{array} \quad \left[ \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

The work is essentially done. It shows that the only solution of the original system is  $(1, 0, -1)$ . However, since there are so many calculations involved, it is a good practice to check the work. To verify that  $(1, 0, -1)$  is a solution, substitute these values into the left side of the original system, and compute:

$$\begin{array}{rcl} 1(1) - 2(0) + 1(-1) & = & 1 - 0 - 1 = 0 \\ 2(0) - 8(-1) & = & 0 + 8 = 8 \\ 5(1) - 5(-1) & = & 5 + 5 = 10 \end{array}$$

The results agree with the right side of the original system, so  $(1, 0, -1)$  is a solution of the system. ■

Example 1 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

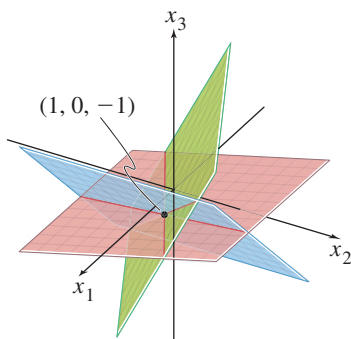
#### ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.<sup>1</sup>
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system. Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

It is important to note that row operations are *reversible*. If two rows are interchanged, they can be returned to their original positions by another interchange. If a

<sup>1</sup> A common paraphrase of row replacement is “Add to one row a multiple of another row.”



Each of the original equations determines a plane in three-dimensional space. The point  $(1, 0, -1)$  lies in all three planes.

row is scaled by a nonzero constant  $c$ , then multiplying the new row by  $1/c$  produces the original row. Finally, consider a replacement operation involving two rows—say, rows 1 and 2—and suppose that  $c$  times row 1 is added to row 2 to produce a new row 2. To “reverse” this operation, add  $-c$  times row 1 to (new) row 2 and obtain the original row 2. See Exercises 29–32 at the end of this section.

At the moment, we are interested in row operations on the augmented matrix of a system of linear equations. Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system. Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. This discussion justifies the following statement.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Though Example 1 is lengthy, you will find that after some practice, the calculations go quickly. Row operations in the text and exercises will usually be extremely easy to perform, allowing you to focus on the underlying concepts. Still, you must learn to perform row operations accurately because they will be used throughout the text.

The rest of this section shows how to use row operations to determine the size of a solution set, without completely solving the linear system.

## Existence and Uniqueness Questions

Section 1.2 will show why a solution set for a linear system contains either no solutions, one solution, or infinitely many solutions. Answers to the following two questions will determine the nature of the solution set for a linear system.

To determine which possibility is true for a particular system, we ask two questions.

### TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

These two questions will appear throughout the text, in many different guises. This section and the next will show how to answer these questions via row operations on the augmented matrix.

**EXAMPLE 2** Determine if the following system is consistent:

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\2x_2 - 8x_3 &= 8 \\5x_1 - 5x_3 &= 10\end{aligned}$$

**SOLUTION** This is the system from Example 1. Suppose that we have performed the row operations necessary to obtain the triangular form

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\x_2 - 4x_3 &= 4 \\x_3 &= -1\end{aligned} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



At this point, we know  $x_3$ . Were we to substitute the value of  $x_3$  into equation 2, we could compute  $x_2$  and hence could determine  $x_1$  from equation 1. So a solution exists; the system is consistent. (In fact,  $x_2$  is uniquely determined by equation 2 since  $x_3$  has only one possible value, and  $x_1$  is therefore uniquely determined by equation 1. So the solution is unique.) ■

**EXAMPLE 3** Determine if the following system is consistent:

$$\begin{aligned}x_2 - 4x_3 &= 8 \\2x_1 - 3x_2 + 2x_3 &= 1 \\4x_1 - 8x_2 + 12x_3 &= 1\end{aligned}\tag{5}$$

**SOLUTION** The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To obtain an  $x_1$  in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

To eliminate the  $4x_1$  term in the third equation, add  $-2$  times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}\tag{6}$$

Next, use the  $x_2$  term in the second equation to eliminate the  $-2x_2$  term from the third equation. Add 2 times row 2 to row 3:

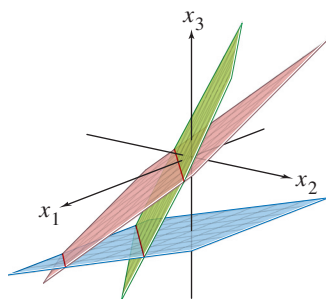
$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}\tag{7}$$

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation:

$$\begin{aligned}2x_1 - 3x_2 + 2x_3 &= 1 \\x_2 - 4x_3 &= 8 \\0 &= 15\end{aligned}\tag{8}$$

The equation  $0 = 15$  is a short form of  $0x_1 + 0x_2 + 0x_3 = 15$ . This system in triangular form obviously has a built-in contradiction. There are no values of  $x_1, x_2, x_3$  that satisfy (8) because the equation  $0 = 15$  is never true. Since (8) and (5) have the same solution set, the original system is inconsistent (i.e., has no solution). ■

Pay close attention to the augmented matrix in (7). Its last row is typical of an inconsistent system in triangular form.



The system is inconsistent because there is no point that lies on all three planes.

### NUMERICAL NOTE

In real-world problems, systems of linear equations are solved by a computer. For a square coefficient matrix, computer programs nearly always use the elimination algorithm given here and in Section 1.2, modified slightly for improved accuracy.

The vast majority of linear algebra problems in business and industry are solved with programs that use *floating point arithmetic*. Numbers are represented as decimals  $\pm.d_1 \cdots d_p \times 10^r$ , where  $r$  is an integer and the number  $p$  of digits to the right of the decimal point is usually between 8 and 16. Arithmetic with such numbers typically is inexact, because the result must be rounded (or truncated) to the number of digits stored. “Roundoff error” is also introduced when a number such as  $1/3$  is entered into the computer, since its decimal representation must be approximated by a finite number of digits. Fortunately, inaccuracies in floating point arithmetic seldom cause problems. The numerical notes in this book will occasionally warn of issues that you may need to consider later in your career.

### PRACTICE PROBLEMS

Throughout the text, practice problems should be attempted before working the exercises. Solutions appear after each exercise set.

1. State in words the next elementary row operation that should be performed on the system in order to solve it. [More than one answer is possible in (a).]

<p>a. <math>x_1 + 4x_2 - 2x_3 + 8x_4 = 12</math></p> <p style="padding-left: 40px;"><math>x_2 - 7x_3 + 2x_4 = -4</math></p> <p style="padding-left: 40px;"><math>5x_3 - x_4 = 7</math></p> <p style="padding-left: 40px;"><math>x_3 + 3x_4 = -5</math></p>	<p>b. <math>x_1 - 3x_2 + 5x_3 - 2x_4 = 0</math></p> <p style="padding-left: 40px;"><math>x_2 + 8x_3 = -4</math></p> <p style="padding-left: 40px;"><math>2x_3 = 3</math></p> <p style="padding-left: 40px;"><math>x_4 = 1</math></p>
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2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.

$$\left[ \begin{array}{cccc} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{array} \right]$$

3. Is  $(3, 4, -2)$  a solution of the following system?

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= -7 \end{aligned}$$

4. For what values of  $h$  and  $k$  is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$